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Extra Derivative Multistep Collocation Methods with Trigonometrically-fitting for Oscillatory Problems

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- Trigonometrically fitting the method
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Objectives

1. To derive extra derivative multistep methods of order 3 and 4, using collocation technique.

2. Trigonometrically-fitting the methods.

3. Validating the methods using a set of special second order differential equations which have oscillatory solutions.

4. Comparing the numerical results with other well known existing methods in the scientific literature.

1. Introduction

The special second order ordinary differential equations (ODEs) can be represented by

$$y" = f(x, y),\ y(x_0) = x_0,\ y'(x_0) = y_0' \qquad (1)$$

in which the first derivative does not appear explicitly.

This type of problems often appears in the field of science, mathematics and engineering such as quantum mechanics, spatial semi-discretizations of wave equations, populations modeling and celestial mechanics.

Currently, equation (1) can be solved directly without converting it first to a system of first order ODEs. Such methods are direct multistep method, Runge-Kutta Nyström (RKN) method, and hybrid method.

Work RKN can be seen can be seen in Dormand et al. [1] and Sommeijer [2]. Work on hybrid method can be seen in Franco [3] and Coleman [4],where they have developed hybrid algorithm and constructed the order condition of the hybrid methods for directly solving equation (1).

Research on directly solving equation (1) using methods which involved, the interpolation and collocation techniques has been done by various researchers such as the work of authors in [5]-[6]. As well as the work of Awoyemi [7] and Jator [8].

The special second order ODEs also often exhibit oscillatory solutions which cannot be solved efficiently using conventional methods.

To obtain a more efficient process for solving oscillatory problems, numerical methods are constructed by taking into account the nature of the problem. This results in methods in which the coefficients depend on the problem.

Numerical methods can adapted to the special structure of the problem, such techniques for this purpose are trigonometrically-fitted and phase-fitted techniques.

A good theoretical foundation of exponentially fitted technique was given by Lyche [9]. Since then a lot of exponentially fitted linear multistep have been constructed mostly for special second order ODEs such as (1).

Vanden Berghe et. al [10]. Introduced an explicit exponentially fitted explicit Runge-Kutta which integrate exactly first order systems whose solution can be expressed as linear combination of functions of the form $\{e^{\lambda t}, e^{-\lambda t}\}$ or $\{\cos(\omega t), \sin(\omega t)\}$.

This idea is extended to Runge-Kutta method by Simos [11] and Franco [12].

Fang and Wu [13] , developed a Trigonometrically fitted explicit Numerov-type method for second-order initial value problems with oscillating solutions.

In this research, we are going to construct extra derivative multistep methods of order 3 and 4 using collocation technique and Chebyshev polynomial will be used as the basis function

In order to improve the efficiency of the methods, we trigonometrically fitted the methods so that the coefficients will depend on the fitted frequency and step size of the problems. And used the methods for solving oscillatory Differential Equations

2. Derivation of Linear Multistep Methods (LMM) Using Collocation Technique

The general k-step LMM for solving special second order ODEs is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \ ,$$

Here, we will use Chebyshev Polynomials as basis function. The following are the first five terms of the sequence from Chebyshev Polynomials :

$$\mathrm{T}_0(x) = 1, \mathrm{T}_1(x) = x, \mathrm{T}_2(x) = 2x^2 - 1, \quad \mathrm{T}_3(x) = 4x^3 - 3x, \mathrm{T}_4(x) = 8x^4 - 8x^2 + 1 \qquad (2)$$

Here, we are going to develop linear multistep method with extra derivatives of the form of

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \mu_j f_{n+j} + h^3 \sum_{j=0}^{k} \eta_j g_{n+j} \ , \qquad (3)$$

where α_j, μ_j and η_j are constant values, $f_{n+j} = y''_{n+j}$ and $g_{n+j} = y'''_{n+j}$.

We proceed to approximate the exact solution $y(x)$ by the interpolating function of the form

$$y(x) = \sum_{j=0}^{n} a_j T_{(n)}(x - x_k)\,, \tag{4}$$

which is the polynomial of degree *n* and satisfied the equations

$$y''(x) = f\big(x, y(x), x_k \le x \le x_{k+p}\big)\ , \qquad y(x_k) = y_k. \tag{5}$$

2.1. Derivation of LMMC(3)

For $n = 4$, equation (4) can be written as

$$y(x) = a_0 + a_1(x - x_k) + a_2[(x - x_k)^2 - 1] + a_3[(x - x_k)^3 - 3(x - x_k)]$$
$$+a_4[(x - x_k)^4 - 8(x - x_k)^2 + 1]. \qquad (6)$$

Differentiating equation (6) three times and we get the first, second, and third derivatives of the equations as follows:

$$y'(x) = a_1 + a_2 2(x - x_k) + a_3 3[(x - x_k)^2 - 1]$$
$$+a_4 4[(x - x_k)^3 - 4(x - x_k)], \qquad (7)$$

$$y''(x) = 2a_2 + a_3 6(x - x_k) + a_4 4[3(x - x_k)^2 - 4], \qquad (8)$$

$$y'''(x) = 6a_3 + a_4 24(x - x_k), \qquad (9)$$

Next, equations (6) and (8) are collocated at $x = x_{k+1}, x_{k+2}$ and interpolated equation (9) at $x = x_{k+1}$ which yields

$$\begin{aligned} y(x_{k+1}) = a_0 + a_1(x_{k+1} - x_k) + a_2[(x_{k+1} - x_k)^2 - 1] \\ + a_3[(x_{k+1} - x_k)^3 - 3(x_{k+1} - x_k)] \\ + a_4[(x_{k+1} - x_k)^4 - 8(x_{k+1} - x_k)^2 + 1] = y_{k+1}, \end{aligned} \tag{10}$$

$$\begin{aligned} y(x_{k+2}) = a_0 + a_1(x_{k+2} - x_k) + a_2[(x_{k+2} - x_k)^2 - 1] \\ + a_3[(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)] \\ + a_4[(x_{k+2} - x_k)^4 - 8(x_{k+2} - x_k)^2 + 1] = y_{k+2}, \end{aligned} \tag{11}$$

$$y''(x_{k+1}) = 2a_2 + a_3 6(x_{k+1} - x_k) + a_4 4[3(x_{k+1} - x_k)^2 - 4] = f_{k+1}, \quad (12)$$

$$y''(x_{k+2}) = 2a_2 + a_3 6(x_{k+2} - x_k) + a_4 4[3(x_{k+2} - x_k)^2 - 4] = f_{k+2}, \quad (13)$$

$$y'''(x_{k+1}) = 6a_3 + a_4 24(x_{k+1} - x_k) = g_{k+1}. \quad (14)$$

By substituting $h = x_{k+1} - x_k$ and $2h = x_{k+2} - x_k$ into equations (10)-(14), we obtain the following:

$$y_{k+1} = a_0 + (h)a_1 + a_2[h^2 - 1] + a_3[h^3 - 3h] + a_4[h^4 - 8h^2 + 1], \quad (15)$$

$$y_{k+2} = a_0 + (2h)a_1 + a_2[4h^2 - 1] + a_3[8h^3 - 6h] + a_4[16h^4 - 32h^2 + 1], \quad (16)$$

$$f_{k+1} = 2a_2 + a_3(6h) + a_4[12h^2 - 16], \quad (17)$$

$$f_{k+2} = 2a_2 + a_3(12h) + a_4[48h^2 - 16], \quad (18)$$

$$g_{k+1} = 6a_3 + a_4(24h). \quad (19)$$

Rearranging equations (15)-(19) into matrix form as follows:

$$\begin{bmatrix} 1 & h & h^2-1 & h^3-3h & h^4-8h^2+1 \\ 1 & 2h & 4h^2-1 & 8h^3-6h & 16h^4-16h^2+1 \\ 0 & 0 & 2h & 6h^2 & 12h^2-16 \\ 0 & 0 & 2h & 12h^2 & 48h^2-16 \\ 0 & 0 & 0 & 6h & 24h^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ hf_{k+1} \\ hf_{k+2} \\ hg_{k+1} \end{bmatrix}$$

which can be simplified as

$$XA = Y \qquad (20)$$

we have

$$A = X^{-1}Y \tag{21}$$

where

$$X^{-1} = \begin{bmatrix} 2 & -1 & \frac{10h^4 - 7}{12h^3} & \frac{2h^4 + 6h^2 + 7}{12h^3} & -\frac{2h^4 + 12h^2 + 7}{12h^2} \\ -\frac{1}{h} & \frac{1}{h} & -\frac{13h^2 - 12}{12h^2} & -\frac{5h^2 + 12}{12h^2} & \frac{3(h^2 + 2)}{12h^2} \\ 0 & 0 & -\frac{2}{3h^3} & \frac{3h^2 + 4}{6h^3} & -\frac{3h^2 + 2}{3h^2} \\ 0 & 0 & \frac{1}{3h^2} & -\frac{1}{3h^2} & \frac{1}{2h} \\ 0 & 0 & -\frac{1}{12h^3} & \frac{1}{12h^3} & -\frac{1}{12h^2} \end{bmatrix}$$

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Solving (21), the coefficients of a_0,a_1, a_2, a_3, and a_4 are obtained in terms of y_{k+1}, y_{k+2} , f_{k+1}, f_{k+2} and g_{k+1}.

$$a_0 = 2y_{k+1} - y_{k+2} + \frac{1}{12}\frac{10h^4 - 7}{h^2}f_{k+1} + \frac{1}{12}\frac{2h^4 + 6h^2 + 7}{h^2}f_{k+2} - \frac{1}{12}\frac{2h^4 + 12h^2 + 7}{h^2}g_{k+1},$$

$$a_1 = -\frac{y_{k+1}}{h} + \frac{y_{k+2}}{h} - \frac{1}{12}\frac{13h^2 - 12}{h}f_{k+1} - \frac{1}{12}\frac{5h^2 + 12}{h}f_{k+2} + \frac{3}{4}(h^2 + 2)g_{k+1},$$

$$a_2 = -\frac{2}{3}\frac{f_{k+1}}{h^2} + \frac{1}{6}\frac{3h^2 + 4}{h^2}f_{k+2} - \frac{1}{3}\frac{(3h^2 + 2)}{h}g_{k+1},$$

$$a_3 = \frac{1}{3}\frac{f_{k+1}}{h} - \frac{1}{3}\frac{f_{k+2}}{h} + \frac{1}{2}g_{k+1},$$

$$a_4 = -\frac{1}{12}\frac{f_{k+1}}{h^2} + \frac{1}{12}\frac{f_{k+2}}{h^2} - \frac{1}{12}\frac{g_{k+1}}{h}.$$

Substituting the coefficients into equation (6) and letting $x = x_{k+3}$, we obtain the following equation:

$$
\begin{aligned}
y(x_{k+3}) = &\left(2y_{k+1} - y_{k+2} + \frac{1}{12}\frac{10h^4 - 7}{h^2}f_{k+1} + \frac{1}{12}\frac{2h^4 + 6h^2 + 7}{h^2}f_{k+2}\right. \\
&\left. - \frac{1}{12}\frac{2h^4 + 12h^2 + 7}{h^2}g_{k+1}\right) \\
&+ \left(-\frac{y_{k+1}}{h} + \frac{y_{k+2}}{h} - \frac{1}{12}\frac{13h^2 - 12}{h}f_{k+1} - \frac{1}{12}\frac{5h^2 + 12}{h}f_{k+2}\right. \\
&\left. + \frac{3}{4}(h^2 + 2)g_{k+1}\right)(x_{k+3} - x_k) \\
&+ \left(-\frac{2}{3}\frac{f_{k+1}}{h^2} + \frac{1}{6}\frac{3h^2 + 4}{h^2}f_{k+2} - \frac{1}{3}\frac{(3h^2 + 2)}{h}g_{k+1}\right)[(x_{k+3} - x_k)^2 - 1] \\
&+ \left(\frac{1}{3}\frac{f_{k+1}}{h} - \frac{1}{3}\frac{f_{k+2}}{h} + \frac{1}{2}g_{k+1}\right)[(x_{k+3} - x_k)^3 - 3(x_{k+3} - x_k)] \\
&+ \left(\frac{1}{3}\frac{f_{k+1}}{h} - \frac{1}{3}\frac{f_{k+2}}{h} + \frac{1}{2}g_{k+1}\right)[(x_{k+3} - x_k)^4 - 8(x_{k+3} - x_k)^2 + 1] = y_{k+3}.
\end{aligned}
$$

Letting $3h = (x_{k+3} - x_k)$, we obtain the discrete form of LMMC as

$$y_{k+3} = 2y_{k+2} - y_{k+1} + \frac{h^2}{6}(-f_{k+1} + 7f_{k+2} - hg_{k+1}), \qquad (22)$$

2.2 Order and Consistency of LMMC method

Definition 1:

The linear difference operator L is defined by

$$L[y(x);h] = \sum_{j=0}^{k}\left[\alpha_j y(x+jh) - h^2\mu_j f(x+jh) - h^3\eta_j g(x+jh)\right],$$

where $y(x)$ is an arbitrary function that is sufficiently differentiable on $[a,b]$. By expanding the test function and its first derivative as Taylor series about x and collecting the terms to obtain

$$L[y(x);h] = c_0 y(x) + c_1 h y'^{(x)} + \cdots + c_q h^{(q)} y^{(q)}(x) + \cdots ,$$

where the coefficients of c_q are constants independent of $y(x)$. In particular

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k} \alpha_j\,, \quad c_1 = \sum_{j=0}^{k} (j\alpha_j)\,, \\
c_2 &= \sum_{j=0}^{k} \left(\frac{j^{(2)}}{2!}\alpha_j - \mu_j\right), \\
c_3 &= \sum_{j=0}^{k} \left(\frac{j^{(3)}}{3!}\alpha_j - j\mu_j - \eta_j\right), \\
&\vdots \\
c_q &= \sum_{j=0}^{k} \left(\frac{j^{(q)}}{q!}\alpha_j - \frac{j^{(q-2)}}{(q-2)!}\mu_j - \frac{j^{(q-3)}}{(q-3)!}\eta_j\right).
\end{aligned}\right\} \qquad (23)$$

Definition 2:

The associated linear multistep method (22) is said to be of the order ρ if

$c_0 = c_1 = \cdots = c_{\rho+1} = 0$ and $c_{\rho+2} \neq 0$.

Definition 3 *[Consistency of the method]*

The method is said to be consistence if it has order at least one.

By substituting the coefficients into equations (23), we obtain
$c_0 = c_1 = c_2 = c_3 = c_4 = 0$ and $c_5 = \frac{1}{8}$.

The new method is consistent and has order $p = 3$.

And denoted as linear multistep method with extra derivative using collocation technique of order three (LMMC(3)).

2.2. Derivation of LMMC(4) , order and consistency of the method

In this section, we derive the LMMC of order four. For $n = 5$, we obtain equation (4) as

$$y(x) = a_0 + a_1(x - x_k) + a_2[(x - x_k)^2 - 1] + a_3[(x - x_k)^3 - 3(x - x_k)]$$
$$+a_4[(x - x_k)^4 - 8(x - x_k)^2 + 1]$$
$$+a_5[(x - x_k)^5 - 20(x - x_k)^3 + 5(x - x_k)]. \qquad (24)$$

Differentiating equation (24) three times gives

Differentiating equation (24) three times gives

$$y'(x) = a_1 + a_2 2(x - x_k) + a_3 3[(x - x_k)^2 - 1]$$
$$+a_4 4[(x - x_k)^3 - 4(x - x_k)] + a_5 5[(x - x_k)^4 - 12(x - x_k)^2 + 1], \qquad (25)$$

$$y''(x) = 2a_2 + a_3 6(x - x_k) + a_4 4[3(x - x_k)^2 - 4]$$
$$+a_5 20[(x - x_k)^3 - 6(x - x_k)], \qquad (26)$$

$$y'''(x) = 6a_3 + a_4 24(x - x_k) + a_5 60[(x - x_k)^2 - 2], \qquad (27)$$

Equations (24) and (27) are collocated at $x = x_{k+1}, x_{k+2}$, and equation (26) at

$x = x_{k+2}, x_{k+3}$ which yields

$$y_{k+1} = a_0 + (h)a_1 + a_2[h^2 - 1] + a_3[h^3 - 3h] + a_4[h^4 - 8h^2 + 1]$$

$$+a_5[h^5 - 20h^2 + 5h] , \tag{28}$$

$$y_{k+2} = a_0 + (2h)a_1 + a_2[4h^2 - 1] + a_3[8h^3 - 6h] + a_4[16h^4 - 32h^2 + 1]$$

$$+a_5[32h^5 - 160h^3 + 10h], \tag{29}$$

$$f_{k+2} = 2a_2 + a_3(6h) + a_4[12h^2 - 16] + a_5[20h^3 - 120h], \tag{30}$$

$$f_{k+3} = 2a_2 + a_3(18h) + a_4[48h^2 - 16] + a_5[540h^3 - 360h], \tag{31}$$

$$g_{k+1} = 6a_3 + a_4(24h) + a_5(60hh^2 - 120), \tag{32}$$

$$g_{k+2} = 6a_3 + a_4(48h) + a_5(240hh^2 - 120). \tag{33}$$

Equations (29)-(33) are written in matrix form as follows:

$XA = Y$

Where

$$X = \begin{bmatrix} 1 & h & h^2-1 & h^3-3h & h^4-8h^2+1 & h^5-20h^3+5h \\ 1 & 2h & 4h^2-1 & 8h^3-6h & 16h^4-32h^2+1 & 32h^5-160h^3+10h \\ 0 & 0 & 2h & 12h^2 & 48h^3-16h & 160h^4-240h^2 \\ 0 & 0 & 2h & 18h^2 & 108h^3-16h & 540h^4-360h^2 \\ 0 & 0 & 0 & 6h & 24h^2 & 60h^3-120h \\ 0 & 0 & 0 & 6h & 48h^2 & 240h^3-120h \end{bmatrix},$$

$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]^T$, and

$Y = [y_{k+1} \quad y_{k+2} \quad hf_{k+2} \quad hf_{k+3} \quad hg_{k+1} \quad hg_{k+2}]^T.$

$$a_0 = 2y_{k+1} - y_{k+2} + \frac{3(6h^4 + 6h^2 + 7)}{20h^2} f_{k+2} + \frac{(2h^4 - 8h^2 - 21)}{20h^2} f_{k+3}$$

$$- \frac{(32h^4 + 72h^2 + 49)}{60h} g_{k+1} - \frac{(17h^4 - 18h^2 - 56)}{30h} g_{k+2},$$

$$a_1 = -\frac{1}{h} y_{k+1} + \frac{1}{h} y_{k+2} - \frac{(153h^4 + 120h^2 + 110)}{100h^3} f_{k+2}$$

$$+ \frac{(3h^4 + 120h^2 + 100)}{100h^3} f_{k+3} + \frac{(392h^4 + 480h^2 + 165)}{300h^2} g_{k+1}$$

$$+ \frac{(149h^4 - 690h^2 - 495)}{300h^2} g_{k+2},$$

$$a_2 = \frac{3(3h^2+4)}{10h^2} f_{k+2} - \frac{2(h^2+3)}{5h^2} f_{k+3} - \frac{2(9h^2+7)}{15h^2} g_{k+1} + \frac{(9h^2+32)}{15h^2} g_{k+2}$$

$$a_3 = -\frac{2(h^2+1)}{5h^2} f_{k+2} + \frac{2(h^2+1)}{5h^2} f_{k+3} + \frac{(8h^2+3)}{15h^2} g_{k+1} - \frac{(23h^2+18)}{30h^2} g_{k+2},$$

$$a_4 = \frac{3}{20h^2} f_{k+2} - \frac{3}{20h^2} f_{k+3} - \frac{7}{60h} g_{k+1} + \frac{4}{15h} g_{k+2},$$

$$a_5 = -\frac{1}{50h^3} f_{k+2} + \frac{1}{50h^3} f_{k+3} + \frac{1}{100h^2} g_{k+1} - \frac{3}{100h^2} g_{k+2}.$$

We substitute a_0, a_1, a_2, a_3, a_4 and a_5 into equation (24) and by letting $x = x_{k+3}$, and $3h = (x_{k+3} - x_k)$ we obtain the discrete form of LMMC as

$$y_{k+3} = 2y_{k+2} - y_{k+1} + \frac{h^2}{10}(9f_{k+2} + f_{k+3}) - \frac{h^3}{30}(g_{k+1} + 2g_{k+2}), \qquad (34)$$

By substituting back the coefficients into equations (23), we obtain

$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$ and $c_6 = -\frac{1}{144}$.

From Definition 2, the new method is consistent and the order is $p = 4$.

And denoted as linear multistep method with extra derivative using collocation technique (LMMC(4)).

3. Trigonometrically-Fitting The Methods

In this section, we adapt the trigonometrically-fitting technique to LMMC(3). By letting some of the coefficients to be unknown values of κ_i, for $i = 1,2,3$, (22) is written as follows:

$$y_{n+1} = 2y_n - y_{n-1} + h^2(\kappa_1 f_{n-1} + \kappa_2 f_n) + h^3(\kappa_3 g_{n-1}). \quad (35)$$

Assuming that $y(x)$ is a linear combination of the functions $\{\sin(vx), \cos(vx)\}$ for $v \in \mathcal{R}$. We obtain the following equations:

$$\left.\begin{aligned} \cos(H) &= 2 - \cos(H) - H^2(\kappa_1 \cos(H) + \kappa_2 + \kappa_3 H \sin(H)), \\ \kappa_1 \sin(H) &= \kappa_3 H \cos(H), \end{aligned}\right\} \quad (36)$$

where $H = vh$, h is the step size and v is fitted frequency.

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Solving equation (36) and letting, $\kappa_1 = -1/6$, the value of the remaining coefficients is obtain in terms of H.

$$\kappa_2 = \frac{7}{6} + \frac{3}{80}H^4 + \frac{851}{60480}H^6 + \frac{20777}{3628800}H^8 + \frac{7939}{3421440}H^{10} + O(H^{12}),$$

$$\kappa_3 = -\frac{1}{6} - \frac{1}{18}H^2 - \frac{1}{45}H^4 - \frac{17}{1890}H^6 - \frac{31}{8505}H^8 - \frac{691}{467775}H^{10} + O(H^{12}).$$

This new method is denoted as Trigonometrically-fitted linear multistep method with extra derivative using collocation technique of order three (TF-LMMC(3)).

Then we apply the trigonometrically-fitting technique to LMMC(4). By letting some of the coefficients to be unknown values of κ_i, for $i = 1,2,3,4$, rewrite the formula in (34) as

$$y_{n+1} = 2y_n - y_{n-1} + h^2(\kappa_1 f_n + \kappa_2 f_{n+1}) + h^3(\kappa_3 g_{n-1} + \kappa_4 g_n). \qquad (37)$$

Assuming that $y(x)$ is the linear combination of the function $\{\sin(vh), \cos(vh)\}$ for $v \in \mathcal{R}$. Therefore, the following equations are obtained.

$$\left.\begin{aligned} \cos(H) &= 2 - \cos(H) - H^2(\kappa_1 + \kappa_2 \cos(H) + \kappa_3 H \sin(H)), \\ \kappa_2 \sin(H) &= -H[\kappa_3 \cos(H) + \kappa_4], \end{aligned}\right\} \qquad (38)$$

where $H = vh$, h is the step size and v is fitted frequency.

Solving equations in (38) simultaneously by letting $\kappa_1 = 9/10$ and $\kappa_3 = -1/30$ the value of the remaining coefficients is obtained in terms of H as follows:

$$\kappa_2 = \frac{1}{10} - \frac{1}{144}H^4 - \frac{313}{100800}H^6 - \frac{923}{725760}H^8 - \frac{6437}{12474000}H^{10} + O(H^{12}),$$

$$\kappa_4 = -\frac{1}{15} + \frac{3}{400}H^4 + \frac{83}{432000}H^6 + \frac{983}{1209600}H^8 + O(H^{10}).$$

This new method is denoted as Trigonometrically-fitted linear multistep method with extra derivative using collocation technique of order four (TF-LMMC(4)).

The other coefficients of the method remain the same. This method is

4. Numerical Results and Discussion

In this section, the new methods LMMC(3), LMMC(4), TF-LMMC(3) and TF-LMMC(4) are tested for problems .1-5 . Comparisons are made with other existing methods.

The following are the notation used in figures 1-10:

TF-LMMC(3)	**Trigonometrically-fitted linear multistep method with collocation method of order three developed in this paper.**
LMMC(3)	A linear multistep method with collocation method of order three developed in this paper.
TF-LMMC(4)	Trigonometrically-fitted linear multistep method with collocation method of order four developed in this paper.
LMMC(4)	A linear multistep method with collocation method of order four developed in this paper.
EHM3(4)	Explicit three-stage fourth-order hybrid method derived by Franco [12]

RKN3(4)	**Explicit three-stage fourth-order RKN method by Hairer et al.[14].**
PFRKN4(4)	Explicit four-stage fourth-order Phase-fitted RKN method by Papadopoulos et al.[15].
DIRKN(HS)	Diagonally implicit three-stage fourth-order RKN method derived in Sommeijer [16]
DIRKN3(4)	Diagonally implicit three-stage fourth-order RKN method derived in Senu et al. [17]
SIHM3(4)	Semi-implicit three-stage fourth-order hybrid method developed in Ahmad et al. [18]

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The following are efficiency curves TF-LMMC(3) Method.

Problem 1*(An almost Periodic Orbit problem studied by Stiefel and Bettis [19])*

$y_1''(x) + y_1(x) = 0.001\cos(x)$,
$y_1(0) = 1, y_1'(0) = 0$,

$y_2''(x) + y_2(x) = 0.001\sin(x)$,
$y_2(0) = 0, y_2'(0) = 0.9995$,

Exact solution is $y_1 = \cos(x) + 0.0005x\sin(x)$, and $y_2 = \sin(x) - 0.0005x\cos(x)$.

The fitted frequency is $v = 1$.

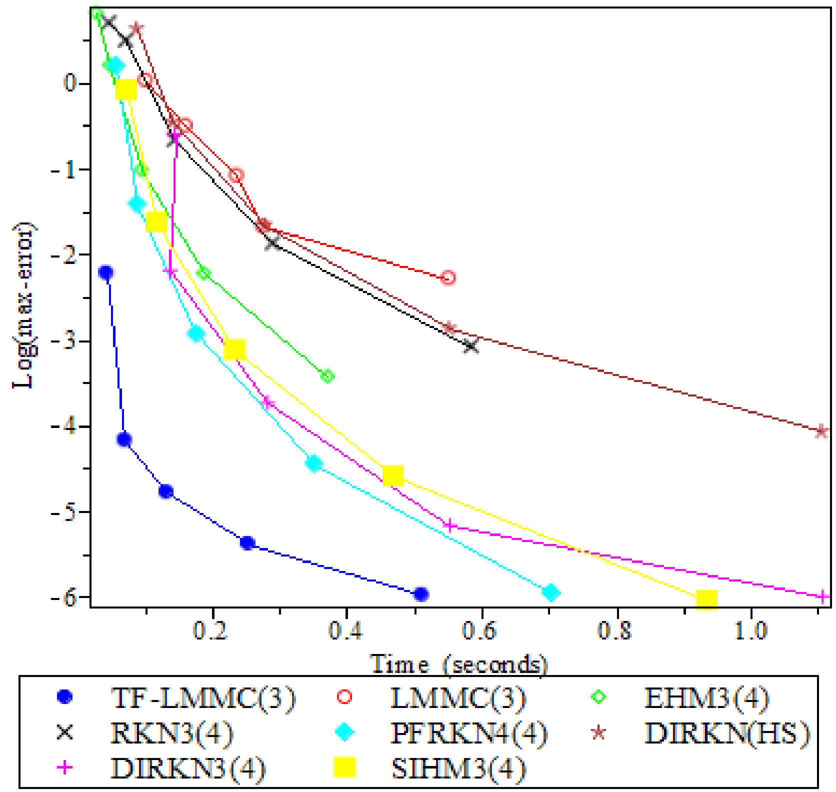

Figure 1: The efficiency curve for TF-LMMC(3) for Problem 1 with $T_{end} = 10^4$ and $h = \frac{0.9}{2^i}$ for $i = 1, \ldots, 5$.

Problem 2 *(Inhomogeneous system by Lambert and Watson [20])*

$$\frac{d^2y_1(x)}{dt^2} = -v^2y_1(x) + v^2f(x) + f''(x),$$

$$y_1(0) = a + f(0), y_1'(0) = f'(0),$$

$$\frac{d^2y_2(x)}{dt^2} = -v^2y_2(x) + v^2f(x) + f''(x)$$

$$y_2(0) = f(0), y_2'(0) = va + f'(0)$$

Exact solution is $y_1(x) = a\cos(vx) + f(x)$, $y_2(x) = a\sin(vx) + f(x)$, $f(x)$ is chosen to be $e^{-0.05x}$ and parameters v and a are 20 and 0.1 respectively.

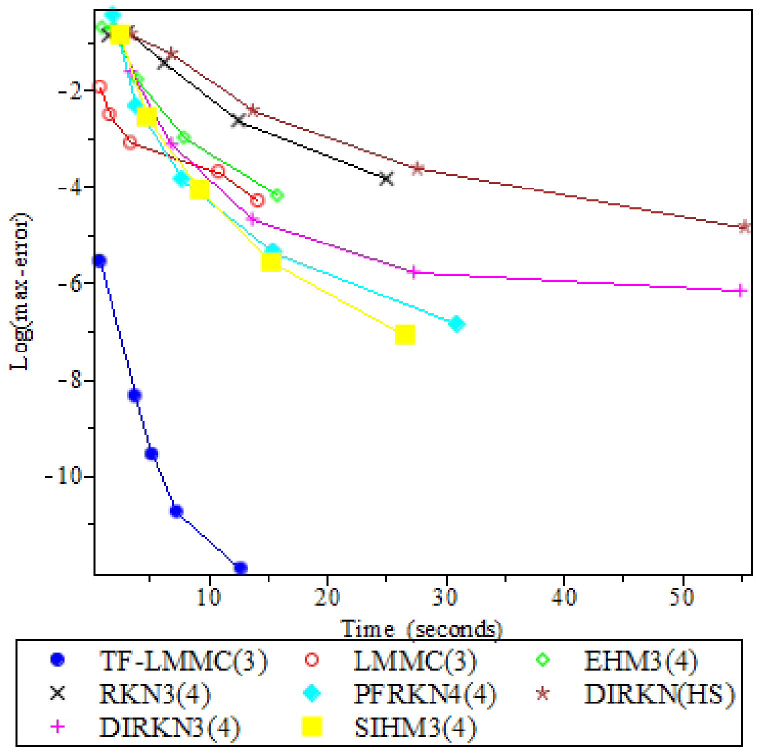

Figure 2: The efficiency curve for TF-LMMC(3) for Problem 2 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 2, \ldots, 6$.

Problem 3 *(Inhomogeneous system studied by Franco [12])*

$$y''(x) = \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix}$$

$$y(x) = \delta \begin{pmatrix} \frac{93}{2}\cos(2x) & -\frac{99}{2}\sin(2x) \\ \frac{93}{2}\sin(2x) & -\frac{99}{2}\cos(2x) \end{pmatrix}$$

$$y(0) = \begin{pmatrix} -1+\delta \\ 1 \end{pmatrix}, y'^{(0)} = \begin{pmatrix} -10 \\ 10+2\delta \end{pmatrix}$$

for $\delta = 10^{-3}$.

Exact solution

$$y(t) = \begin{pmatrix} -\cos(10x) - \sin(10x) + \delta\cos(2x) \\ \cos(10x) + \sin(10x) + \delta\sin(2x) \end{pmatrix}.$$

The Eigen-value of the problem are $v = 10$ and $v = 1$.

The fitted frequency is chosen to be $v = 10$ because it is dominant than $v = 1$.

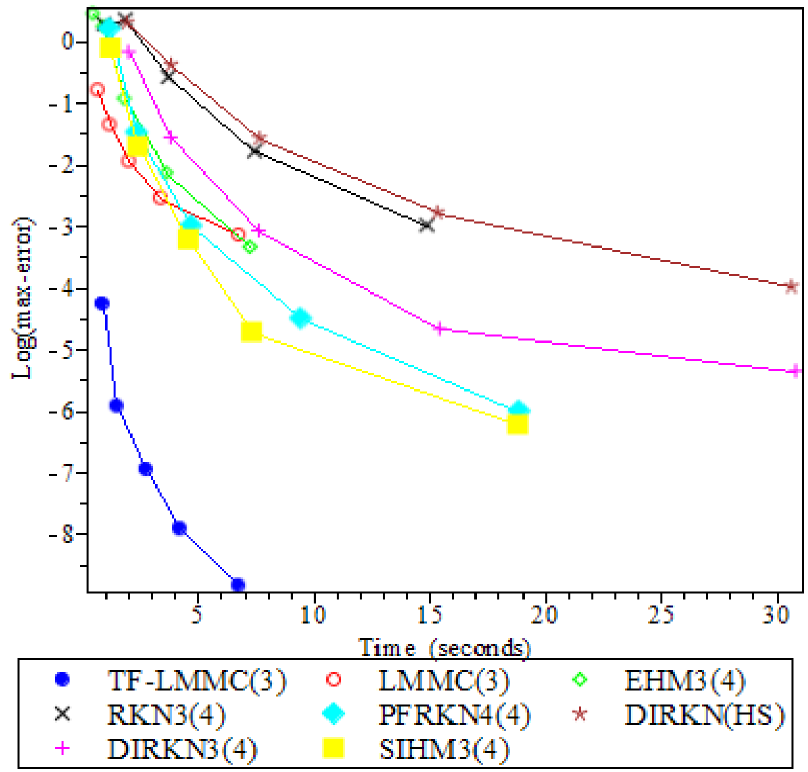

Figure 3: The efficiency curve for TF-LMMC(3) for Problem 3 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 1, \ldots, 5$.

Problem 4 *(Homogenous given in Attili et.al [2])*

$y''(x) = -64y(x)$,$y(0) = {}^1/_4$,

$y'(0) = -{}^1/_2$.

Exact solution is $y = {}^{\sqrt{17}}/_{16} \sin(8x + \theta)$,

$\theta = \pi - \tan^{-1}(4)$.

The fitted frequency is $v = 8$.

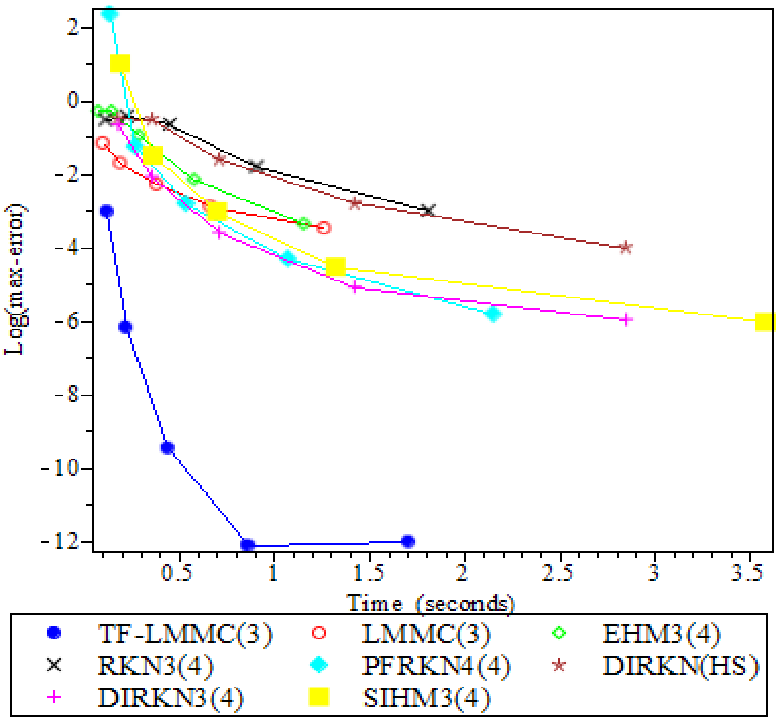

Figure 4: The efficiency curve for TF-LMMC(3) for Problem 5 with $T_{end} = 10^4$ and $h = \frac{0.1}{2^i}$ for $i = 3, \ldots, 7$.

Problem 5 *(Inhomogeneous equation studied by Papadopoulos et.al [15])*

$y''(x) = -v^2 y(x) + (v^2 - 1)\sin(x),$

$y(0) = 1, y'(0) = v + 1.$

Exact solution is $y(x) = \cos(vx) + \sin(vx) + \sin(x).$

The fitted frequency is $v = 10.$

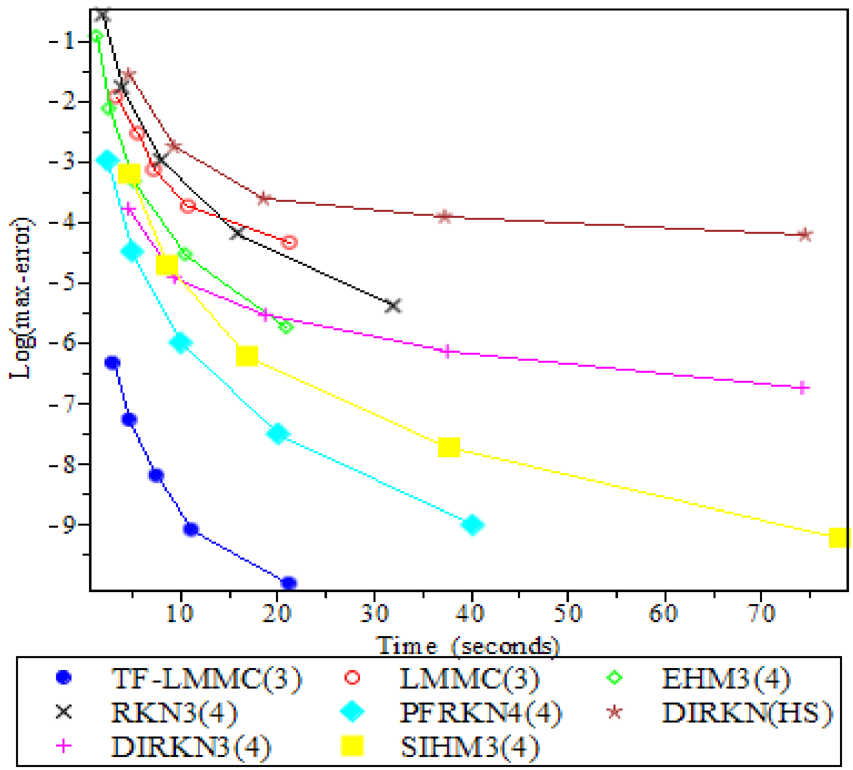

Figure 6: The efficiency curve for TF-LMMC(3) for Problem 6 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 3, \dots, 7$.

The efficiency curves are shown in Figures 1-5, where problems 1-5 are tested for a very large interval $T_{end} = 10000$.

It is observed that TF-LMMC(3) lies below all of the other methods efficiency graphs. It prove that TF-LMMC(3) is superior compared to the other existing methods in the literature.

LMMC(3) is as competitive as the other methods for solving oscillatory problems, though LMMC(3) is method of order three which is one order less compared to the other methods in comparison.

Although explicit methods such as EHM3(4) and RKN3(4) needs less time to do the computation, the methods are less efficient compared to the other implicit and fitted methods.

Implicit methods: DIRKN(HS), DIRKN3(4), SIHM3(4) and phase-fitted method: PFRKN4(4), need more time to do the computation, thus less efficient compared to TF-LMMC(3). For all of the problems tested,

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The following are efficiency curves TF-LMMC(4) Method.

Problem 1*(An almost Periodic Orbit problem studied by Stiefel and Bettis [19])*

$y_1''(x) + y_1(x) = 0.001\cos(x)$,
$y_1(0) = 1, y_1'(0) = 0$,

$y_2''(x) + y_2(x) = 0.001\sin(x)$,
$y_2(0) = 0, y_2'(0) = 0.9995$,

Exact solution is $y_1 = \cos(x) + 0.0005x\sin(x)$, and $y_2 = \sin(x) - 0.0005x\cos(x)$.

The fitted frequency is $v = 1$.

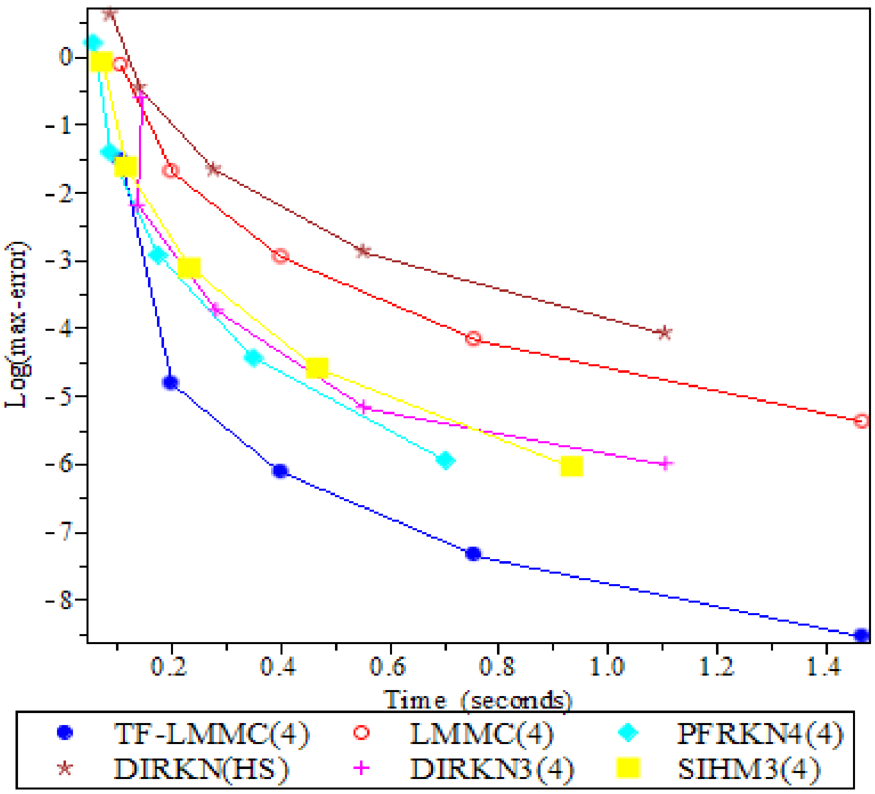

Figure 6: The efficiency curve for TF-LMMC(4) for Problem 1 with $T_{end} = 10^4$ and $h = \frac{0.9}{2^i}$ for $i = 1, \ldots, 5$.

Problem 2 *(Inhomogeneous system by Lambert and Watson [20])*

$$\frac{d^2y_1(x)}{dt^2} = -v^2y_1(x) + v^2f(x) + f''(x),$$

$$y_1(0) = a + f(0), y_1'(0) = f'(0),$$

$$\frac{d^2y_2(x)}{dt^2} = -v^2y_2(x) + v^2f(x) + f''(x)$$

$$y_2(0) = f(0), y_2'(0) = va + f'(0)$$

Exact solution is $y_1(x) = a\cos(vx) + f(x)$, $y_2(x) = a\sin(vx) + f(x)$, $f(x)$ is chosen to be $e^{-0.05x}$ and parameters v and a are 20 and 0.1 respectively.

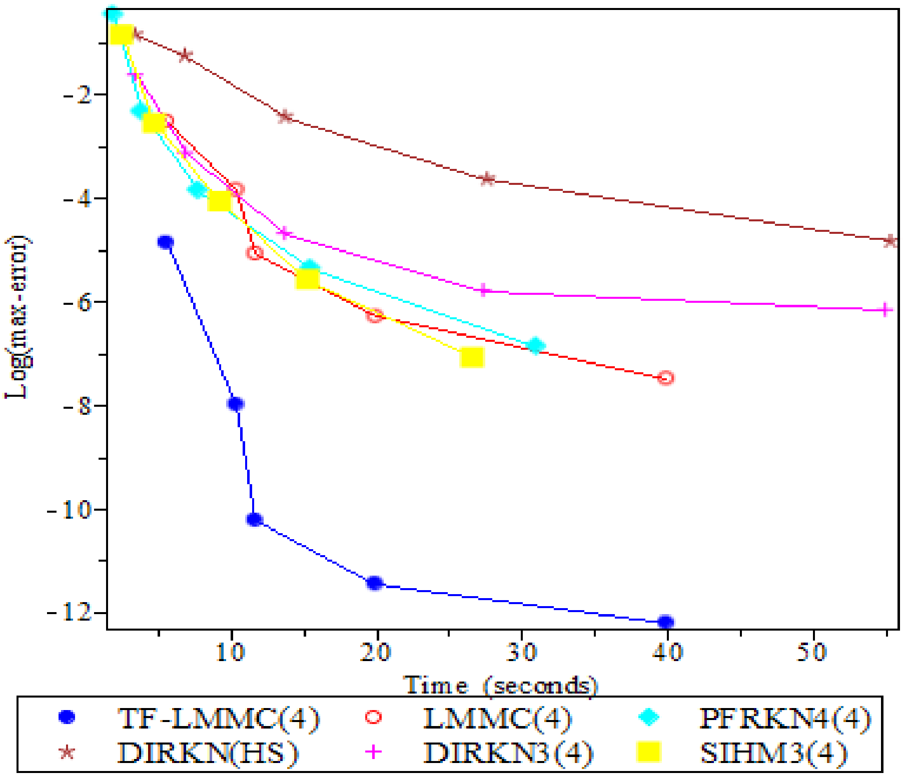

Figure8: The efficiency curve for TF-LMMC(4) for Problem 2 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 2, \ldots, 6$.

Problem 3 *(Inhomogeneous system studied by Franco [12])*

$$y''(x) = \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix}$$

$$y(x) = \delta \begin{pmatrix} \frac{93}{2}\cos(2x) & -\frac{99}{2}\sin(2x) \\ \frac{93}{2}\sin(2x) & -\frac{99}{2}\cos(2x) \end{pmatrix}$$

$$y(0) = \begin{pmatrix} -1+\delta \\ 1 \end{pmatrix}, y'^{(0)} = \begin{pmatrix} -10 \\ 10+2\delta \end{pmatrix}$$

for $\delta = 10^{-3}$.

Exact solution

$$y(t) = \begin{pmatrix} -\cos(10x) - \sin(10x) + \delta\cos(2x) \\ \cos(10x) + \sin(10x) + \delta\sin(2x) \end{pmatrix}.$$

The Eigen-value of the problem are $v = 10$ and $v = 1$.

The fitted frequency is chosen to be $v = 10$ because it is dominant than $v = 1$.

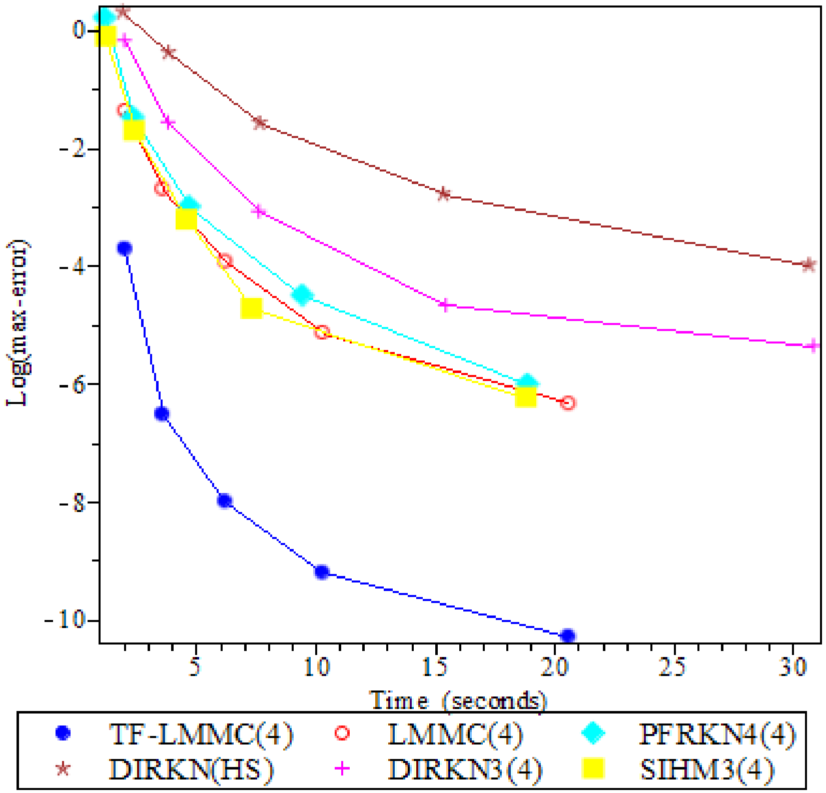

Figure 9: The efficiency curve for TF-LMMC(4) for Problem 3 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 1, \dots, 5$.

Problem 4 *(Homogenous given in Attili et.al [2])*

$y''(x) = -64y(x)$,$y(0) = {}^{1}/_{4}$,
$y'(0) = -{}^{1}/_{2}$.

Exact solution is $y = {}^{\sqrt{17}}/_{16} \sin(8x + \theta)$,
$\theta = \pi - \tan^{-1}(4)$.

The fitted frequency is $v = 8$.

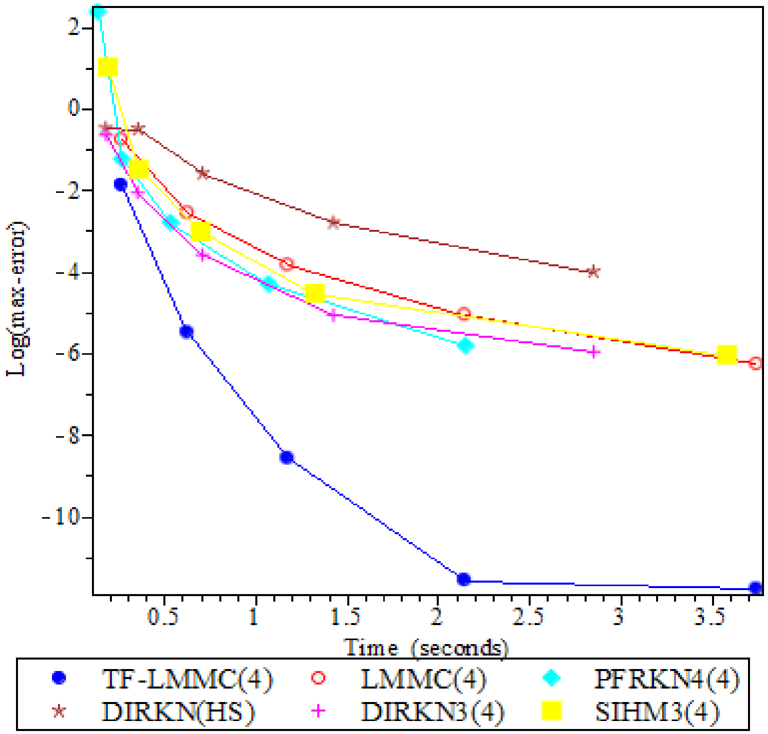

Figure 9: The efficiency curve for TF-LMMC(4) for Problem 4 with $T_{end} = 10^4$ and $h = \frac{0.1}{2^i}$ for $i = 3, \dots, 7$.

Problem 5 *(Inhomogeneous equation studied by Papadopoulos et.al [15])*

$y''(x) = -\nu^2 y(x) + (\nu^2 - 1)\sin(x),$

$y(0) = 1, y'(0) = \nu + 1.$

Exact solution is $y(x) = \cos(\nu x) + \sin(\nu x) + \sin(x).$

The fitted frequency is $\nu = 10.$

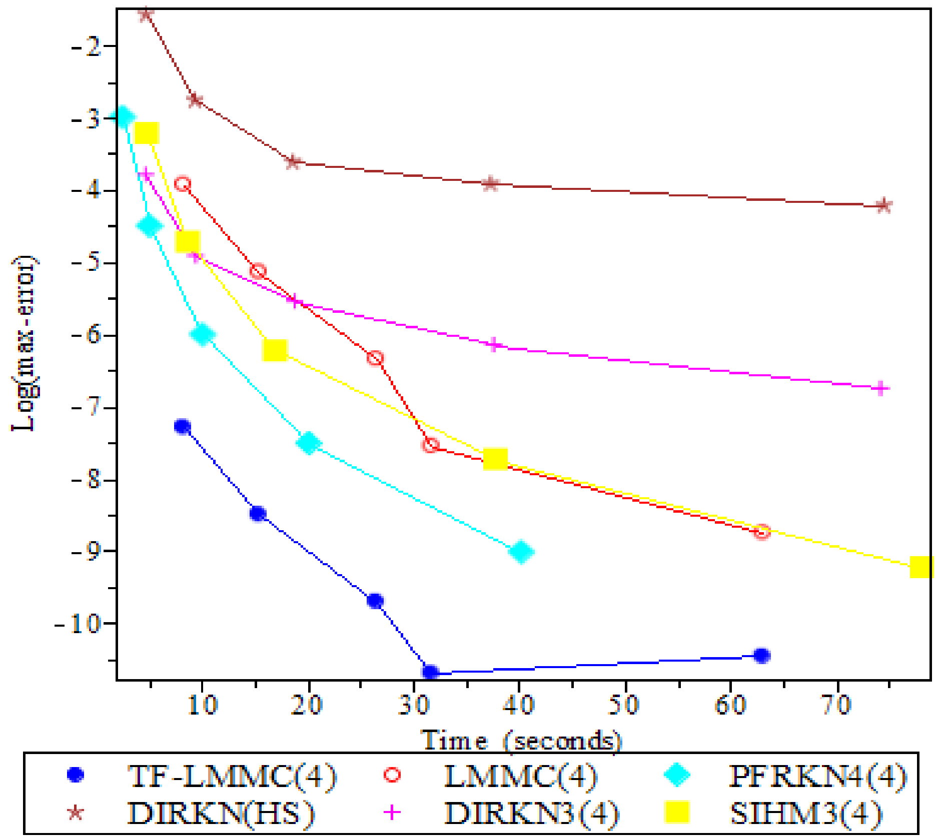

Figure 10: The efficiency curve for TF-LMMC(4) for Problem 5 with $T_{end} = 10^4$ and $h = \frac{0.125}{2^i}$ for $i = 3, \ldots, 7$.

From the efficiency curves in Figures 6-10, we observed that methods with fitting properties have smaller error.

DIRKN methods have more functions evaluation than LMMs and SIHMs, hence more computational time is required to implement DIRKN methods.

It is shown that TF-LMMC(4) have better performance compared to the original method LMMC(4), and superior compared to all the existing methods in comparisons.

Conclusion

- In this research, we developed linear multistep methods with **extra derivatives** using collocation technique of order three (LMMC(3)) and four (LMMC(4))

- **Trigonometrically-Fitted** the Linear Multistep Method With Collocation technique of Order Three (TF-LMMC(3)) and four (TF-LMMC(4)) respectively.

- Numerical results for LMMC(3) which has order three is as comparable as other existing methods which are of order four
- Numerical results for LMMC(4) which is order four is slightly better than other existing methods in comparisons.
- Hence having extra derivtives in the multistep method do improved the accuracy of the methods.
- TF-LMMC(3) and TF-LMMC(4) are clearly superior in solving special second order ODEs with oscillatory solutions since it involves lesser computational time and better accuracy
- We can conclude that Trigonometrically-fitting the methods improved the effiency of the methods for integrating oscillatary problems.

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